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THE MATHEMATICS TEACHER

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ROMANCE OF MATHEMATICS

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It may seem unwarranted to speak of romance in connection with a subject commonly supposed to be as dry and prosaic as mathematics. The dictionary defines a romance as a "fictitious and wonderful tale." The tale which I am about to relate is indeed *wonderful* but it is not fictitious—it is true and it is the kind of truth that is stranger than fiction. It is the story of the birth and growth of the number concept in the human race—it is the story of the struggle of humanity to learn to count. We speak glibly of "thousands," and "millions," and "billions," but we should not forget that these concepts are the cumulative heritage of many centuries. Even to-day there are backward tribes in some parts of the world who have no number words beyond "one," "two," or "three" and who designate all larger groups under the single word "many."

The aboriginal Indian tribes of North America afford excellent illustrations of the primitive difficulties in learning to count. They had no number *names* and no number *symbols* such as we now possess. They used the names of the fingers for one, two, three, four, five, and the phrases "one on the other hand," "two on the other hand," etc., for six, seven, and so on, up to ten which they called "half an Indian." For eleven they used the phrase "one down," meaning one of the toes, for twelve "two down," etc., up to fifteen. For sixteen "one on the other foot," for twenty "whole Indian," twenty-one "one on the other Indian," for forty "two Indians," etc. Try to imagine if you can what we should do to-day with a number system like that and then try to comprehend the age-long struggle by which our perfected numerical machinery was evolved from

such crude beginnings. The oldest known mathematical document in the world, which was written by an Egyptian priest about 1700 years before Christ on a roll of papyrus now preserved in the British Museum, shows certain crude beginnings of arithmetical computation. Hence we may actually trace the development of the number concept back through at least three to four thousand years, and we may be sure that its ultimate beginnings were coincident with the very dawn of human intelligence. Just as the babe in arms knows the difference between "one" and "three" when one of its toys is taken away from it, so the primitive human race began the struggle of *learning to count* even before it *learned to talk*.

What little we know of the history of mathematics during the thousand or two thousand years preceding 600 B.C. is mostly due to the Egyptians. For the next 1200 years to A.D. 600 the Greeks added glory to their *other* intellectual achievements by greatly advancing the knowledge of mathematics, especially of geometry under Euclid and of algebra under Diophantus. Then came the dark ages from 600 to 1500 A.D. when all learning in Europe sank to a dead level of mediocrity; but fortunately during this period the Hindus and the Arabians kept alive and advanced the knowledge of mathematics to a higher degree than had before been attained. The Arabians in the rôle of world conquerors brought this knowledge into Spain in the eighth century, where it was kept alive in the Moorish schools for 400 years. In the thirteenth century Leonardo of Pisa travelled and studied in Spain and he wrote a book showing the great superiority of Arabian mathematics over anything then known in Europe, but it was not until the sixteenth century that the revival of learning in Europe really began and not until the seventeenth, eighteenth, and nineteenth centuries that mathematics attained its full development in Europe as a system of thought worthy of the highest consideration. Those were the centuries which produced intellectual giants such as Descartes, the inventor of analytic geometry, Newton, the inventor of the calculus, Leibniz, La Grange, Laplace, Legendre, Euler, Gauss, Dedekind, Weierstrass, and so on through scores of names all of which will remain forever written high on the roll of fame. Thus the fortunes of mathematics, as far back as we can trace them, rested first in the East, chiefly in Egypt, for a thousand

years, then in Greece for twelve hundred years, then in India and Arabia for a thousand years, and finally in Western Europe for five hundred years, where it attained its great development.

This progress of mathematics, although painfully slow, through three or four thousand years, has followed closely the expansion of the idea of number in the human mind. *The story of the number system is the romance of mathematics.*

What, then, *is* the number concept? We speak of five apples, five dollars, five miles, but what is *five?* It is a *mental creation,* an abstraction quite apart from any word, such as apples, dollars, or miles—it is the one distinguishing common property of these groups which are equivalent not as to color, or size, or weight, or shape, but only as to *plurality,* and this equivalence determines what we call the *cardinal number* of the group.

These cardinal numbers are whole numbers or integers. They were the only numbers known to primitive tribes. It requires a much higher degree of mental development to comprehend the abstract notion of a fraction, and the aborigines had very little use for such abstractions. The Egyptians and the early Greeks considered only unit fractions, that is, those with the numerator one. It was not until 2000 years after the Egyptian priest wrote his famous papyrus that common fractions came into general use. Fractions have always been difficult for the human race and are still for many individuals of the race.

A still greater difficulty arose with reference to *negative numbers.* They are so familiar to all of us now that it does not seem possible that such a concept could have held up the progress of mathematics for three thousand years, but such is the fact. The Egyptians carefully avoided negative numbers. The Hindus recognized them in solving equations but would not admit them as solutions. They said that such solutions were "inadequate" and that people "did not approve them." The Arabs refused to admit them and did not bring them into Spain. They were noticed by Cardan who solved the cubic equation in the sixteenth century, but it remained for Descartes in the seventeenth century to show that negative numbers are just as important as positive numbers and he did this by the simple device of numbering the points on an unlimited straight line, which can only be done by starting at some point and numbering in *both* directions, since the line has no end point with which to

begin; and hence the two series of numbers came to be distinguished as positive and negative. This simple invention is the basis of analytic geometry and of all modern graphic methods such as are now used in every kind of statistical work. This invention did for mathematics what the Edison mazda electric lamps have done for a world previously lighted with tallow candles.

Descartes also did another great service in giving respectable standing to the number *zero* which properly designates the starting point on the numbered line, but which also does untold service for the decimal notation in giving *place value* to the digits which precede and follow it. For us who use the decimal notation as we would some perfect mechanical device, which may have had a long and difficult development in order to reach its perfection, it is difficult to understand how the world got along for some thousands of years with such clumsy devices for representing numbers as were used in Europe prior to the thirteenth century. The Greeks used twenty-seven letters for number symbols and had a most complicated system of notation largely because they had no zero to indicate place value. The Romans had fewer symbols and quite a scientific system but no zero. You will appreciate the seriousness of this lack if you attempt to multiply two numbers, such as 252 and 785, in the Roman notation. It was the Hindus and the Arabs who perfected the decimal notation as well as our present number symbols, including the zero, and these were brought into Europe by way of Spain, but it took nearly 300 years for Europe to appreciate fully this great contribution of the East to the West—in some respects, perhaps, the most important invention the world has ever seen. Imagine if you can the development of the highly complicated mathematical theory which underlies the applications of electricity if the clumsy number notations and symbols of mediæval Europe were in use to-day throughout the world. The very structure of modern civilization rests upon the perfected system of mathematics made possible by the Hindu-Arabic decimal system and number symbols.

Not only was a perfected number system essential to a mathematical structure, but also symbols of operation were needed. The plus and minus signs were invented by the Germans in 1489, the multiplication and equality signs by the English in

1530, the division sign by the Germans in 1659, exponents by the French in 1540, letters for unknown quantities by the French and English in 1629. Up to the fifteenth century all mathematical documents had been written out in words either in full or partly abbreviated, but with no symbols such as are now known.

But a mere collection of number words and number symbols cannot be called a *system* unless it is organized on the basis of a set of laws or rules governing its structure. Such laws began to be formulated and were brought to perfection in Western Europe in the seventeenth, eighteenth, and nineteenth centuries, and this intensive study of the number concept gave rise to some of the most difficult problems of higher mathematics and challenged the keenest thinking of the intellectual giants of those centuries.

For instance, the question whether the square root of 2 should be considered as a number arose in the minds of the Hindus in the twelfth century. They recognized that the square root of 2 is not a whole number and that it is not a common fraction and since these were all the kinds of numbers then known, they naturally debarred such a quantity from their number system. This skeptical attitude existed in Europe and continued for four hundred years when the question again arose in connection with Cardan's solution of the cubic equation. It was recalled also that Pythagoras two thousand years earlier had shown that the length of the diagonal of a unit square must be the square root of 2. From the seventeenth century on for two hundred years the nature of these so-called *irrational numbers* like $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{5}$, etc., was the subject of the most profound investigation, till finally in the nineteenth century it was shown that such quantities obey the same laws as the whole numbers and common fractions and hence have the same right to be included in the number system.

A still more difficult question arose in connection with so-called *imaginary numbers* like, for example, the square root of negative 4. Cardan encountered such quantities also in solving the cubic equation and he said they were "curious" but "useless." Little could he have dreamed that those same "curious" quantities were to form the basis of the most recondite and most powerful mathematical machinery that the human brain has

ever devised. These quantities, later called *complex numbers*, were subjected to three hundred years of intensive study and finally in the nineteenth century they were admitted to full fellowship in the number system because it was then shown that they could and would obey the same laws as the other numbers.

You will recall Descartes's great invention wherein he represented the positive and negative whole numbers and zero as *sentinels* on the unit points of a straight line with the common fractions in endless succession between these points. When the irrational numbers were admitted to the number system, they also were found to represent points on the line and, moreover, when these were all in place, behold the line was full and every point was occupied, so that there was no room for the complex numbers. However, several mathematicians of the nineteenth century, including Argand, Gauss and Wessel, invented a graphic representation, called the Argand diagram, or the Gauss Complex Plane, for the complex numbers, which was an achievement even more wonderful than what Descartes did for the real numbers. They showed that every point in the plane could be occupied by a complex number as a sentinel and that all the other numbers on Descartes's number line were only special cases of complex numbers.

We have then seen that our present number system, starting some four or five thousand years ago with a few whole numbers, has now grown to contain a countless succession of whole numbers in both the positive and negative directions, separated by zero, and interspersed with a countless succession of fractions and irrational numbers between each two successive whole numbers, and finally the doubly countless array of the complex numbers covering all the plane.

What, then, shall we say of this Number System the romantic history of which we have traced through the age-long struggle of the human race to learn to count?

1. In the first place this system is *complete*. There are no more numbers possible to be included under the laws which govern the operations of our arithmetic and algebra.

2. In the second place this number system is *continuous*. As you pass along Descartes's number line or wander over Gauss's complex plane you do not *jump* from one number to another—but the change is like a *continuously flowing* stream.

This property of the number system is the foundation on which Sir Isaac Newton built the calculus and it is one of the most profound achievements of all time.

3. In the third place, this number system is *adequate* for the solution of the world's great problems. It underlies the whole fabric of modern civilization. We have long known that mechanics, physics, engineering and all such physical developments are based on mathematics, but we have only more recently realized that our knowledge of the whole domain of electricity (which in view of the modern electron theory of matter may dominate the entire universe) rests on mathematical doctrine of the highest order. Chemistry, likewise, is dependent on mathematics, and we are now discovering that research workers in nearly every scientific field are unable to proceed beyond certain definite barriers without a working knowledge of mathematics. A well-known illustration of this fact is the remarkable service which the great Steinmetz rendered to the General Electric Company and through them to the world by perfecting the highly complicated mathematical theory of the alternating current, which, strange to say, rests upon the theory of the complex numbers.

4. In the fourth place this number system and the vast mathematical doctrine growing out of it are *alive* and *working* to-day, contrary to the popular notion that mathematics is a dry, dead shell. Moreover, there is ample evidence that the leadership in mathematical research in the twentieth century may be transferred from Europe to America. Great strides have been made in this direction during the first quarter of this century and no little credit is due to Johns Hopkins, Harvard and Princeton Universities in the East and to the University of Chicago in the West for their leadership in this line.

5. Finally, I venture to assert that the perfected number system, with all its implications and developments, is in many respects the *greatest achievement of the human race:* greatest in *difficulty* as shown by the struggle of the race through four thousand years to achieve it; greatest in its *scientific implications* as shown by the far-reaching investigations to which it has given rise; and greatest in usefulness as a servant of mankind, as shown by its underlying relation to practically every other science.

POSTULATES AND SEQUENCE IN EUCLID

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Almost any proposition in geometry is implied by another, already accepted; that may have been implied by still others, but in this regression we finally reach statements that appear not to be dependent upon implication. They are the postulates or definitions upon which all the science of geometry is based—the "first causes." They are originally the result of analyzing the world outside our minds, to clarify the ideas of the simplest objects in it and of their simplest relations. Some such primary statements can by taking thought be shown to be implied by others. In framing a geometrical system, as Euclid did, it is necessary to choose which of the results of our analysis shall be our postulates or definitions. The choice usually made for elementary geometry to-day is in good part the choice made by Euclid.[1] We can, however, be more explicit than he, particularly in regard to the idea of what we call number.

Two Kinds of Postulates.—Modern writers make a distinction between axioms and postulates which does not seem to be warranted by the words themselves. Historically they are synonyms. To express the one idea there were actually four different words used in Euclid's time without any recognized distinction. They were *axioma,* a thing claimed, *aitema,* a thing demanded, *lambanomenon,* a thing admitted, and *hypokeimenon,* a thing supposed or assumed. Euclid's postulates he called *aitemata,* and what his modern editors call axioms he called *koinai ennoiai,* or "common notions."

According to an ancient tradition, this separate listing is said to mean that the "common notions" were assumed by Euclid not for geometry only, but for things in general. That might have been a good distinction to make, but Euclid did not make it. The last two of the "common notions," the axiom of congruence

[1] For an indication of his shortcomings, see J. W. Young, *Fundamental Concepts of Algebra and Geometry* (Macmillan, 1911), p. 12.

and the axiom of inclusion, are clearly geometrical in character. I think the distinction he had in mind, for he surely had one, was between geometry of what we might call a qualitative sort, the idea of shape rather than of size, and the sort of geometry that had to do with number, the idea of measurement. The postulates (*aitemata*) of Book I gave the fundamental facts about the limited straight line, the prolongation of such a line, the circle and the pair of parallels; the postulate "all right angles are equal" appears to have been placed here to be a foundation for the postulate of parallels. The Common Notions, the "axioms" that enabled him to establish a correspondence between number and geometric magnitude, were as follows:

1. If $a = b$, and $b = c$, then $a = c$.
2. If $a = x$, and $b = y$, then $a + b = x + y$.
3. If $a = x$, and $b = y$, then $a - b = x - y$.

4. *The axiom of congruence:* Things which coincide with one another are equal.

5. *The axiom of inclusion:* The whole is greater than the part.

Of these axioms the fourth is the basis of measurement, and the fifth was then and is now the means of approach for magnitudes not integrally measurable in terms of the unit. The other three are his axioms of algebra, enabling him to use his lines for symbols of number. The whole five formed a foundation for arguments based on measurement. It may be that the word "common" was applied to these notions for the reason that measurement and reckoning based upon it were practical and familiar, appropriate to the base uses of trade and becoming truly respectable only when pursued "for the sake of knowledge."[2] On the other hand speculations about the details of shape were the occupation of philosophers, and ordinary people might well be mystified by such things as the postulate of parallels. The matter must be left in this undecided state. We are not even sure that Euclid did divide his postulates and "common notions" as they are here shown. Whether he did or not, his claiming—or demanding—them, meant only that he had no way of demonstrating them. Upon them he based not only his geometry, but his exhaustive discussion of scientific arithmetic.

[2] Plato, *Republic*, 525D (τοῦ γνωρίζειν ἕνεκα).

Some Imperfections.—If we look to these ten statements for postulates which are entirely sufficient for the work of Book I, and at the same time entirely necessary for it, we shall look in vain. Only within the last half century have such postulates been formulated.[3] At the same time it is interesting to see, as examples of Euclid's philosophic insight, anticipations of such modern abstractions as the distributive, commutative, and associative laws.[4] It seems hard to realize that these subtleties, so recently known to us by name, have actually lain concealed in Euclid's text for all these centuries; and even more astonishing that, on account of his Definition of Equal Ratios, they could be applied and were applied to his irrationals.

Bends in the Unbending Tradition.—There is a popular impression that in the old days the basis of geometry was fixed, and the impulse to tamper with it has come only to modern renegades. On the contrary, even one of Euclid's immediate successors felt free to pick his own postulates. Archimedes assumes that a straight line is shorter than any other line having the same extremities, while Euclid thought it desirable to prove that the base of a triangle was shorter than both the other sides taken together. Euclid's proof was based upon the axiom of inclusion; Archimedes wanted to compare curves with straight lines, and the axiom of inclusion was of no use to him. Much later, but long before your grandfather's day, Legendre knocked the *Elements* to pieces and fitted them together according to his own plan. The new system was thereupon protected by the same old taboo. The successive "Books" of the new arrangement were referred to by number, like Euclid's, as fields of thought marked out once for all. There was even a time, under Napoleon III, when the Minister of Public Instruction would have liked to see the same page of Caesar and the same theorem of Legendre recited on at the same hour throughout the realm of France. Playfair in the early years of the nineteenth century revived an old substitute for Euclid's parallel postulate,

[3] Young, *Fundamental Concepts*, Lectures XIV and XV. Of the formulations given there the nearest to current teaching practice is that of Mario Pieri, based upon a neglected textbook, *Elementare Geometrie auf neuer Grundlage streng deduktiv dargestellt*, by J. K. Becker (Berlin, 1877). See also *Abstract Geometry in Terms of Inclusion*, by E. V. Huntington, in *Math. Annalen*, LXXIII, 1913, pp. 522–559.

[4] *Euclid*, II, 1, V, 22, V, 23.

which Euclid himself is supposed to have considered and rejected; namely, that not more than one parallel can be drawn to a straight line through a point without. We have it yet.

With all these tinkerings, there has persisted up to the actual present an impression that those equal right angles of Postulate 4 should still be given a high seat at the table. Sometimes their equality is assumed, as in Euclid, but often it is curiously proved. Euclid needed these equal right angles for his parallel postulate; but for us, using Playfair's postulate, it is quite as sound and much more convenient to prove that our unit angles (angles of one degree) are all equal, and thus to obtain immediate validity for the ordinary measurement of an angle. If that is mensuration, make the most of it.

Euclid's Basis of Sequence.—It is often stated that Euclid's geometry is purposely aloof from mensuration. Nowhere in it will you find a rule for the area of a rectangle or a triangle.[5] Such a conclusion is hardly safe. The expression "area contained by two straight lines" was in itself almost as much a numerical term to the Greeks as the word square. All the purely numerical facts of Book II, which were in good part the rules for everyday computation, are stated in geometrical terms; the product is a rectangle, equal factors produce a square. Even integral products were distinguished as square or oblong.[6] Consequently it was for their purposes entirely sufficient to prove that a triangle was half a parallelogram, and the parallelogram equal to a rectangle. When it comes to rules for the indirect measurement of length, there are plenty of theorems, such as the famous 47th of Book I. For all these proofs the "common notions" were logically necessary.

If we really want to know where Euclid was going with this mathematical system of his, one way to find out would be to see what he arrived at. The last Book, the thirteenth, is devoted to formulating expressions for the lengths of the edges of the regular polyhedra. It refers especially to the first, sixth, and eleventh Books, which are geometrical rather than algebraic, and to the fifth and tenth, which are exclusively algebraic. It does not refer to the twelfth Book, so that we may expect to find there further evidence of purpose, another separate task

[5] H. B. Fine, *Annals of Mathematics*, Vol. 19, p. 76.

[6] Plato, *Theætetus*, 147E.

achieved; and in it we do find the collection of theorems about the volumes of pyramids and cones upon which Archimedes in after years based the results that were the pride of his heart. They all go back to the "rectangular solid," the volume of which is the "solid number" referred to by Euclid in Definition 17, Book VII. In Book XII we are dealing with actual rules for the mensuration of solids, as practical men had known them from the time of Democritus; but the discussion of them is such as Plato could approve—the invention of Eudoxus, and part of the general theory of number and magnitude.

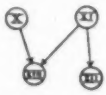
The Three Entrances.—It is significant that Euclid begins in three different places to lay the foundations of his logical structure. Book I, Book V, and Book VII are each entirely self-dependent; any one of them could have been the first of the thirteen Books. Book V as a beginning would certainly have led to a more coherent presentation. It is the theory of proportion.

Book VII and the next two Books, which are devoted exclusively to the theory of integers, could have been made a separate treatise. Their connection with the other Books is very slight. Book VIII makes use only of the definitions of duplicate and triplicate ratio from Book V; one proposition of Book IX, the fifteenth, depends on two formulas of Book II;[7] two or three of the simpler propositions of Book VII, and two or three of those of Book IX, are used in Book X for discussing the rational parts of what we call binomial surds. These three Books, then, sandwiched between the geometry of rational lines and the combination of irrationals and solid geometry, served practically no purpose other than to complete the theory of arithmetic as it was then known to science. There are theorems in the lot that have proofs still entitled to respect. One of them is that the number of primes is infinite, and another that the number $2^n(1+2+4+\cdots+2^n)$, the parenthesis being prime, is a "perfect" number (a number equal to the sum of all its factors). They are not forgotten, and have not been improved; but there is no way to use them in the farm or the counting house, and we consequently do not often hear of them.

The Last Four Books.—If Euclid had begun with Book V, or even with Book VII, instead of with Book I, Book XI would still have been necessary for the last two Books; for it gives the

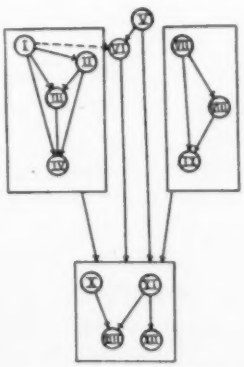
[7] $(a+b)a=ab+a^2$ and $(a+b)^2=a^2+b^2+2ab$.

basic theorems about parallel and perpendicular lines and planes, solid angles, parallelepipeds, and, finally, the theory of the measurement of parallelepipeds and prisms. Without these we could do nothing with the solids of Book XII (pyramids and cones), or with the more complicated theorems of Book XIII. Book X is necessary for Book XIII, and since it would not be reasonable to have it intrude between Book XI and its twin successors, the necessary sequence of these Books would be as in this diagram.



Tracing the Sequence Back.—Books I and V are essential for all of these, and though the few propositions of Book VII and Book IX that are referred to in discussing the rational constituents of the irrationals of Book X could have been introduced into that Book where they were needed, they were kept with their fellows in "the three arithmetical Books" because the whole doctrine of integers was in itself worth while. Then those coherent Books had to precede Book X.

The geometry and the algebra of the first and second Books were independent of the difficulty of incommensurables. Up to Book V these theorems were carried on as far as possible without raising that question. Then the algebra of incommensurables



was developed, and applied in Book VI to the completion of the topics of Book I. Here, in Book VI, is the subject of similar triangles, and here the algebraic device called the Application of Areas is completed. Thus the logical succession of the Books was determined as indicated in the following diagram.

The extent of dependence in each of these successions is roughly indicated in the following table. Each Book as entered in the left hand column depends on the Books listed in the same line on the right. When the dependence is upon isolated propositions, those are separately indicated.

II—I
III—I; II, *5, 6*
IV—I; II, *11;* III
V—
VI—I; III, *27, 31;* V
VII—
VIII—V, *Def.;* VII
IX—II, *3, 4;* VII; VIII
X—I, *44, 47;* II; III, *31;* V; VI; VII, *4, 11, 26;* IX, *1, 24, 26*
XI—I; III, *31;* IV, *1;* V; VI
XII—I; III; IV, *6, 7;* V; VI; X, *1;* XI
XIII—I; II, *4;* III; IV; V; VI; X; XI

Postulates Postponed.—After Book I, for which Euclid has chosen his postulates, no others are needed until we come to the Books that are themselves new and separate foundations. In Book V there is a "definition" which expresses one postulate and implies another. Here it is:

Book V, Definition 4. *Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another.*

This says, in effect, that any two magnitudes of the same kind have a ratio. It applies not only to commensurable magnitudes, which have a ratio that can be expressed in figures, but also to incommensurable magnitudes, which he later proves have not a ratio so expressible. In modern phrase, it would say that every magnitude has a measurement number. This "definition" also implies that of two magnitudes of the same kind either can be so multiplied as to exceed the other. This is a form of the axiom by means of which Eudoxus established the theorems about pyramids and cones. It is known as the "Axiom of Archimedes," having been used by him (with due credit to Eudoxus) in the following form:

Of unequal lines, surfaces, and solids, the greater exceeds the less by such a magnitude as when added to itself can be made

to exceed any given magnitude which has a ratio to the first magnitudes.[8]

It is clear, then, that this definition, as Euclid calls it, is really a postulate (or an axiom, if you prefer that name for it). One of the immediate applications of it is found in the first demonstration in Book X, which reads as follows:

If from the greater of two given magnitudes there be subtracted a magnitude greater than its half, and from the remainder a magnitude greater than its half, and so on continually, there will finally be left some magnitude less than the lesser magnitude given.

This theorem is used first in Book X to show that the ratio of two incommensurable magnitudes can be determined within a given measure of precision; and is used again in Book XII, where by the "method of exhaustion" the famous theorems of Eudoxus, referred to above, are ingeniously proved. Such geometrical implications are of great interest and of historical importance, but the extraordinary thing about this "definition 4" is that it made a formal assertion of the existence of incommensurable numbers (ratios) so long ago.

Standing on the Wrong Foot.—It is not strange that in recasting Euclid for school use the three arithmetical Books should be ignored, and the entire remainder considered as belonging to geometry. It is even understandable that for many generations we should have taken Book II as a similar but more bearable intrusion, and that Book X with its consequences in Book XIII should be omitted as too abstruse. No reason appears, however, for not beginning with the subject of proportion. With Euclid proportions served the purposes of algebraic equations. It was for his pupils necessary to construct an elaborate symbolic vocabulary to express the equations and the operations upon them. Even for us, whether we call them equations or proportions, there is difficulty so far as they are involved with the discussion of incommensurables; but everything is clear enough as soon as incommensurables are either understood, or, as in algebra and trigonometry, ignored.

Dodging a Difficulty.—Now the surprising thing about Eu-

[8] The last clause is Ver Eecke's translation of τῶν πρὸς ἄλληλα λεγομένων. Heiberg translates it *quae cum ea comparari possint*, and Heath gives it as "of the same kind."

clid's exposition of proportion is that the understanding of incommensurables was not involved in it. Beyond saying, in substance, that any two magnitudes of the same kind had a ratio, and then giving a rule, a Test of Multiples, by which it could be decided that the ratio of one pair of magnitudes was the same as the ratio of another pair, his Book V said not a word that had any special reference to incommensurables or to irrationals. These are defined in Book X, and discussed in Books X and XIII. What we call his geometry was independent of that discussion. To be sure, his technique of proportion is repellent; like Holmes's katydid, he says such undisputed things in such a solemn way. But that is not what has bothered us. We did not understand what his Test of Multiples meant, and dreaded to put it at the roots of the subject, for fear it had an unseen attachment on some forgotten twig. Consequently, while reconstruction was more or less in fashion, nobody wanted to disturb this magic abracadabra that worked so well. In more recent times, when his basic definition was seen to contain the same profound significance that modern mathematical philosophy had only recently arrived at, it was regarded as too difficult to explain to beginners.

In our familiar textbooks several attempts have been made, makeshifts most of them, to explain clearly the meaning of a ratio of incommensurables; but we have now, apparently, arrived at the decision that the topic should be postponed. Well, Euclid did just that. He explained later what incommensurables are, as we do. But as to their ratios, he just gave this rule. Probably it was not so mysterious to the Greeks as it seems on our printed pages. Roughly speaking, it meant to them that two ratios would be equal if equal approximations could be found for them within any arbitrary measure of precision. So expressed, it is clear enough to us, and possesses precisely the same scientific validity as Euclid's cumbrous phrasing.

There is no reason why we should not, if we choose, begin geometry by a study of proportion. We are more likely to profit, however, by a study of separate ratios; and in particular, by a study of the ratio of any given magnitude to the unit of that kind of magnitude—the ratio that we call the measurement number of the given magnitude. We need the equivalent of those two definitions of Euclid—the fourth and the fifth of Book V; they can be stated as follows:

(I) *Every distance, area, or volume has a measurement number.*

(II) *Two measurement numbers are equal if equal approximations can be found for them within whatever limits of precision may be arbitrarily prescribed.*

It would be entirely sound, in logic if not in pedagogy, to place these two assumptions first in our geometrical system; we might even give a more usable statement of the second one, as follows:

Whenever a theorem is true about approximations to a geometrical figure, it is true also, on one condition only, about the figure itself; this one condition is that such approximations can be shown to exist within whatever limits of precision may be arbitrarily specified.

Though this assumption—axiom, definition, or whatever you may call it—is of the same logical character as Euclid's Definition of Equal Ratios, and is in itself more easily acceptable by our pupils because of its consonance with their naive notions of measurement, it is probably more useful in the teacher's mind than in the mouths of glib beginners. The best way would probably be to show that the theorems are true for approximations made in routine, no matter how close those approximations could be either actually made or even only imagined; and to point out that since approximations are all we ever have in practice, our conclusions can be relied on even for the most minute precision. With the phrase "or even imagined," all leaks are stopped.

No additional postulates are required for the introduction of trigonometry as an organic part of the logical sequence. We should merely be using in a systematic fashion the powerful theory of similarity, and giving names to certain simple and easily identified classes of ratios.

What Becomes of the Sequence?—Nobody has ever counted the possible rearrangements of the geometrical sequence; and probably few things more unfortunate could happen than to have some particular succession so charm the academic eye as to be established by decree, or otherwise to secure age-long deference. It would be equally unfortunate to regard the range of topics as necessarily fixed. It has been much extended since Euclid's time, but now the tired schoolmaster loves it as an

armchair to rest in. The heirloom that has been so long in the family should not be disturbed or replaced by anything of new fashion. Yet even pedantic orthodoxy must see that the extensions already made, old as they are, were due to the broadening scope of the studies for which the *Elements* were written as preparatory. Geometry, including its algebra, is still preparatory. Mathematics beyond this stage still needs *Elements*, but not necessarily the same *Elements* that Archimedes needed, or even Legendre. Not the same list from now on, and not the same order of succession.

Euclid, and to a lesser degree Legendre, had their writings so fossilized. Never again. Geometry belongs, not to the books written about it yesterday, but to the people who are thinking about it to-day. In it we show and test the weapons by which wide empires of truth have been and are being conquered. Here the future paladins of that conquest prepare for victory, and here even mere spectators may spy out the promised land or hear, each in his own language, messages invaluable. We are teaching a science, for its subject matter and for its method; we are disclosing to our youth the beginnings of the most comprehensive generalizations that the mind of man has ever conceived. Fixed routes and boundaries are unnatural and enslaving. Neither in content nor in sequence should there be any permanent prescription.

AN EARLIER PLACE FOR THE CALCULUS IN THE CURRICULUM

By Professor MARTIN NORDGAARD

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The history of the mathematical curriculum is an interesting study. In colonial Harvard and Yale, algebra, geometry, and trigonometry were college disciplines; Latin and Greek were required for entrance; science had little place. At present, algebra and geometry are high school studies; Latin and Greek are often begun in college; the natural and social sciences are taken from high school freshman to college senior classes. A hundred years ago secondary school mathematics meant chiefly arithmetic, including such difficult topics as alligation, continued fractions, and circulating decimals. It was later realized that the student would gain more practical values and experience greater intellectual pleasure by omitting the more difficult portions of arithmetic, and in their place take the simpler parts of algebra and geometry; the new curriculum harmonized better with the student's capacity, adapted itself better to his experience, and formed a better basis for the courses in science that began to make their appearance in the college curriculum.

Soon algebra and geometry were regularly assigned to the preparatory schools; plane and spherical trigonometry were moved down to the Freshman and Sophomore years, and the newer courses in analytics and calculus were offered in the Junior and Senior years. In time, trigonometry also made its way into the academies, and analytics was offered to the freshman; only rarely has analytics been offered as a separate study in the secondary schools; but a large portion of its subject matter is now taken up as graphical algebra in the senior high schools.

What about the Calculus? Some decades ago if given at all it was a Junior-Senior study. But it has gradually worked its way downward. Being a tool and a foundation for technical courses, engineering colleges have placed the calculus as early in the curriculum as has hitherto seemed possible, namely, in the

sophomore year. In the liberal arts colleges the calculus was until recently regarded as a purely disciplinary subject, and was generally placed in the sophomore and junior years.

But a change is taking place in educated peoples' attitude towards the calculus in regard to its cultural and practical values, and this changing attitude may make it advisable both to alter its place in the curriculum and to modify its manner of presentation. For increasingly the teachers of college courses in physics, mechanics, statistics, finance, and economics wish to use the calculus in their work.

Increasingly, in after-life, the educationist, the biologist, the personnel manager uses the graph and the formula as everyday tools. And many of them experience the need for the calculus also. In a conversation with a successful doctor of medicine in New York City some years ago, he told me it seemed impossible for him to follow up the latest researches in medicine, for the discussion of them in the medical journals was given so largely in the language of the calculus. To overcome this handicap he had set out to plough through a little book that most of you know. Thompson's *Calculus Made Easy*. With calculus, as with any branch of learning, its practical and cultural value must be measured not by whether one can get along without it if one does not know it, but by whether and how much of it one will use if one does know it.

In view of these facts, and since so many students take only one year of college mathematics, is it not advisable to leave out some of the more abstruse portions of trigonometry, college algebra, and analytics, like Sturm's theorem, line functions, and spherical triangles, and take up the concepts of rate, slope, continuity, maxima and minima as developed in an introductory course in the calculus? Many educators think so. Of the many institutions which have tried or are trying this arrangement, I may mention, as typical, University of Maine, University of Montana, University of Illinois, Massachusetts Institute of Technology, Antioch College, Barnard College, the College of the City of New York, Smith College, Ohio State University, Michigan State College, and some colleges whose representatives are in this convention.

One of the earliest ventures into this field was performed, strangely enough, at an engineering school. About twenty years

ago, Professor Woods and Baily, at Massachusetts Institute of Technology, collaborated a unified Freshman text in college algebra, analytics, and the differential calculus. (Trigonometry was required for entrance at M. I. T.) This text was at first widely adopted among colleges; but its program was too difficult and too comprehensive, and the text has fallen into disuse. Other texts for the same purpose sprang up, among which may be mentioned Smith and Gales' *Elementary Analysis* (1910) and McClenon's and Rusk's *Analysis of Elementary Functions* (1918). Five years ago, 1921, appeared a text designed for Freshmen that bids fair to become epochal in college teaching, namely, Griffin's *Introduction to Mathematical Analysis.* It is a most skillfully graded and carefully unified course in college algebra, trigonometry, analytics, and the simpler processes of the differential and integral calculus. Among college texts it is well nigh unique in that the student learns nearly all the routine and technique by solving actual problems from physics, mechanics, economics, statistics, finance, and engineering. Griffin has adopted what is best in the European curriculum of Freshman grade; and he has incorporated a judicious amount of proposed American reforms. The book is already a widely adopted text.

Most attempts at placing an introduction to the calculus in the Freshman year have been made by aid of unified courses in mathematics. In some ways this has been unfortunate, for many teachers are prejudiced against unified mathematics. However, elementary calculus need not be given as part of a unified course, though that undoubtedly economizes time. And so we are grateful for a new text, just published by Harper and Brothers, wherein the introductory calculus is treated separate from trigonometry and analytics. This text is N. J. Lennes' *A Survey Course in Mathematics.*

With the increasing practice of mathematics of reaching out into real life for its problems and illustrations, and its greater emphasis on the function concept; and with the growing interest of people in and out of school in the study of statistics, investment, and the natural sciences, it does not seem that the movement for introducing elementary calculus in the Freshman year will become less. The question is whether this interesting and practical branch of mathematics will not be taken up earlier,

say in the last year of the high school. Do not its foundational, cultural, and practical values, which speak for its inclusion in the Freshman curriculum, hold equally well in respect to the high school senior? And are they not doubly worthwhile, if he does *not* go off to college and has no more opportunity to study mathematics? To answer this question we must take into consideration three things: (1) Has a high school senior the age and the mentality and the experience to profit by such a course? (2) Must essential and vital parts of the high school curriculum be sacrificed in order to take up the elementary calculus? (3) Will the number and scholastic attainments of the teaching force permit it?

As to the last two questions: If the elementary calculus is offered as an elective, there will be no necessity for slighting any fundamental subject in the high school course. But it will give the student with natural aptitude for mathematics and the one who will wish it for future use, the opportunity to learn this science. As to the teaching staff, it goes without saying that only schools with well-trained instructors can afford to offer this course.

The question of age and mentality is the most important. And on this phase I have no need to speak alone. When the report of the National Committee on Mathematical Requirements was published in 1923 its most startling recommendation was that advocating a semester of the elementary calculus as an elective in the senior year of high school. This committee, working under the auspices of the Mathematical Association of America from 1916 to 1923, has produced a most monumental and authoritative work, and their recommendations are based upon wide and painstaking investigations both here and abroad. The report states that in Germany, Austria, and France the calculus is taken in the student's twelfth school year—that is, when he is about seventeen to eighteen years of age. The same is done in Belgium, Denmark, Sweden and Switzerland. In England it is offered even earlier. True enough: thanks to better-trained teachers and a well-planned curriculum, the European student in his twelfth year is on a par with the American college Sophomore in respect to scholastic attainments. But that does not affect his native capacity. And an improved teaching staff and course of study is an ideal toward which to work. *Let the mathematics people do the starting!*

We may get some light on the movement in which we are agents by knowing something of its inception and early development. (*The Mathematical Gazette*, VII, p. 8.) At a School Conference in Berlin in 1900, Professor Klein and others urged that the calculus be included in the *real-gymnasium;* but no action was taken by the Ministry of Education. At a meeting in Breslau in 1904, Professor Klein claimed "that the function idea, graphically represented, should form the central notion of mathematical teaching, and, as a natural consequence, the elements of the calculus should be included in the curriculum of all 9-class schools"—a procedure already proved to be possible in France. The ninth year of the gymnasium is the twelfth school year. Of the twenty-six 9-class schools reporting in 1908, thirteen had introduced the function idea and the calculus. All of them agreed that the experiment was a success and wished to continue. In 1908 the International Congress of Mathematicians met at Rome, and thereafter the reforms in mathematical teaching in continental Europe began to make themselves felt in British and American schools.

In February, 1911, C. S. Jackson read a paper before the London Branch of the Mathematical Association (*Gazette*, VII, 197) on the problem of teaching higher mathematics to the average student. Herein he advocated the teaching of the calculus to boys of the public schools, of ages sixteen and seventeen, and recommends a plan of presentation employing graphical algebra and finite differences as preliminary to the more analytical processes.

In 1912 the International Congress of Mathematicians met in Cambridge. Elementary calculus was again on the agenda, and interest in it was growing. In England especially the idea was gaining ground that the boy who *was not going to the university* and yet had a talent for mathematics should be given a chance to get instruction in the calculus, even at the expense of manipulative skill in algebra and the more formal side of geometry. In a report by a Secondary Schools Special Committee (*Gazette*, VII, p. 231) we find these statements: "They desire to call attention to the fact that a great majority of the boys leave school before they reach the age of seventeen, and that much mathematical work has to be omitted, or unduly postponed, which would be useful to them at school in science

and other subjects, as well as in after life." "It is desirable that boys of ordinary ability who attend a secondary school to the age of sixteen should not leave without some introduction to the principles on which the calculus is based."

In the 1913 General Mathematical Syllabus for pupils not specializing in mathematics (*Gazette,* VII, p. 129) drawn up by the Public Schools Special Committee and approved by the General Committee of the Mathematical Association, it was proposed to permit boys of the fifth and sixth forms (*i.e.,* boys of sixteen to eighteen years of age) to choose elections from Mechanics, Calculus, Analytics, and continuation courses in algebra, geometry, and trigonometry. It was further suggested that those whose mathematical abilities are slight would profit most by taking up elementary mechanics; while next in order of difficulty they suggest the elementary calculus. In Godfrey and Siddon's *Elementary Algebra,* published that same year, we find processes of differentiation and integration worked right in with the functions and graphs. In a detailed report from the Borough Polytechnic Institute (*Gazette,* VII, p. 201) in 1913, we find the plans suggested by C. S. Jackson two years before and the Syllabus of 1913 systematically carried out. In this school the general age of entry is thirteen years and the course lasts three years. The elementary calculus is taken up in the third year. At the end of the year they are able to differentiate the simpler exponential and logarithmic expressions and to integrate such expressions as $\int \cos^2 \theta d\theta$.

Beginning with the International Congress of Mathematicians in Cambridge in 1912, the movement took firm root in England and soon spread to the Scottish schools. The most powerful stimulus was that given by the engineers, as represented by Professor Perry. The physicist had long pressed for a modicum of the calculus in the secondary schools and preferred to have it given without too much mathematical rigor. It was not adopted without opposition, however. There were many who feared the effect on algebraic technique and the formal side of geometry. Others, chiefly professional mathematicians, disliked very much the lack of rigor and logical completeness that characterized the course.

In England no bureaucratic ministry of education could install the new course as was done in France, Germany, and

Austria. English schools have local autonomy and the new course had to run the gauntlet of professional and popular discussion. And so must it do in America before, and if, it is adopted. For our schools, as those of Great Britain, are under local control. The question is one meriting consideration and discussion.

In France, England, Germany, this movement started in university circles; the teachers and boards of the secondary schools helped appraise and evaluate. Also in this country whatever discussions there have been on this point have proceeded largely from college and university men. In fact, the whole question of reorganizing our mathematical curriculum in the grades and the high schools has received its most valuable contribution from a committee selected by our own Mathematical Association; namely, the report of the National Committee on Mathematical Requirements.

I have already intimated that if an introductory course in the calculus is given to college Freshmen or high school seniors, it needs a presentation suitable to those classes. For one thing, too many topics should not be given; by the use of the algebraic polynomial we can initiate the student into the calculus mode of thought and give him the mechanics of differentiation and integration. (Quoting the Report of the National Committee, p. 41.) "The calculus of the polynomial is so simple that a boy or girl who is capable of grasping the idea of limit, of slope, of velocity, may in a brief time gain an outlook upon the field of mechanics and other exact sciences, and acquire a fair degree of facility in using one of the most powerful tools of mathematics, together with the capacity for solving a number of interesting problems. Moreover, the fundamental ideas involved, quite aside from their technical applications, will provide valuable training in understanding and analyzing quantitative relations —and such training is of value to everyone."

A FEW CLASS–ROOM DEVICES TO STIMULATE INTEREST IN MATHEMATICS

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In the writer's capacity as supervisor of mathematics in the Fresno High School, many interesting plans and methods in the class-room have been brought to her attention; and it is the purpose of this paper to recount a few of the devices which teachers and writer have used on various occasions to stimulate an interest in mathematics.

More than half of the twelve hundred students in the senior high school are enrolled in classes in mathematics. There is only one class in elementary algebra, as this subject is provided for in junior high school. But it might be said in passing that the efficiency of the course has been greatly enhanced since the pupils have used one of the splendid exercise tablets, now on the market, with its various timed tests.

The classes in plane geometry are necessarily large, because this subject is an entrance requirement of most colleges. The fact that many students, besides those who wish to follow engineering and scientific courses, are choosing mathematics in their junior and senior years proves that elective courses in mathematics are holding their own with other elective courses in the high school, and that the pupils are finding them both interesting and worth while.

The teachers realize that not only the content of the subject must be such that it will meet the needs and abilities of the students, but their method of presentation must be such that it will hold and sustain the interests of the students. While the method of instruction varies with the individual teacher, each one is conscious of the general aims of the course and is constantly drawing upon the mathematical knowledge and experience of the pupils to furnish material for practical and concrete illustrations.

No doubt every teacher of mathematics has at times felt his ability challenged to keep the interest of the students at a high point; therefore, the writer has decided to suggest to others some of the "interest raising" devices which have proved helpful to the teachers of Fresno.

The past year one teacher secured satisfying results from an experimental class in laboratory plane geometry. The class was conducted to some extent on the "Dalton Plan." At least, the pupils were permitted "freedom to do their own work at their own speed," if not "in their own time." The manual used was well adapted for individualized instruction. The teacher made her assignments in such a way as to encourage the pupils to work according to their individual levels of capacity. In a short time, the class was broken up into groups of various degrees of attainment.

The room in which the class was held was well suited for this kind of experiment. Being equipped with tables and chairs, drawing boards and T-squares, it lent itself not only to laboratory work but also to informal group discussion. Upon visiting the class one thought of a three-ring circus. There might be several pupils at the board in an animated discussion over the solution of some original; another group, with the teacher as the central figure, giving instructions and directions, might be working on laboratory plates with a suggestion here or a correction there; and a third group might be participating in a socialized recitation, demonstrating the fact that "the most effective teaching is self-teaching." In addition, there would be the pupils who needed no assistance, but were working ahead on special problems or maximum assignments.

The minimum essentials were completed by all the pupils, and a large percent of the class made an effort to do part, if not all, of the maximum assignments. Two members of this class conducted an experiment in which they approximated very closely the latitude of Fresno. They made two observations, one on Polaris and the other on the sun. They worked up their experiment in such good shape that they were given the opportunity to discuss their results with some of the other classes. Much interest was aroused over this project, and other pupils were encouraged to attempt something on their own initiative.

Occasionally, some of the teachers guided the review of problems and exercises over a certain unit of the work so as to take on the nature of a contest. On the day before the contest, two leaders were selected who, in turn, chose the other members of the class. The leaders wrote the assigned problems and exercises on uniform slips of paper, and, on the day of the contest,

gave them to the teacher who supervised the drawing. The leaders called from their sides those who were to be the first and second contestants. These, in turn, drew slips from the pile, put the figure of their problem on the board and wrote out the "given" and "to prove." The first one ready stated his problem, explained his figure and proceeded with the demonstration. Those of the class not reciting at the time did the scoring. The class set the score of ten for a correct demonstration. They gave two points for the figure, one point for the "given," one point for "to prove," one point for construction, and five points for the proof. At the end of the demonstration, corrections and criticisms were made. There was little time lost in arriving at a score satisfactory to both sides, since the teacher decided in case of controversy. In order that the contest might be continuous, each team kept two performers at the board. When one was through the other was ready to begin. At the end of the recitation, the scores of the several contestants on each side were added, and the side scoring the highest total number of points was declared the winner.

Such contests put all the pupils on their mettle. At all times the attention was almost one hundred percent. Feeling the keen competition, the performer was aroused to do his best. He had pride not only in his own work, but also in the mass action of his side. Those serving in the capacity of scorers were alert to everything that was said. They weighed the validity of this statement and of that statement, they formed judgments and discriminated between the fine points of a proof. These contests proved to be not merely "interest getting" devices, but also methods of instruction fraught with educational value.

In another scheme for review which one teacher used advantageously, the pupils prepared original true-and-false questions covering the unit of work assigned for the review recitation. At the class time these questions were redistributed, and, at random, pupils were called upon to read their questions and to defend their answers. Sometimes only an axiom, a definition or a construction was required; while again it was necessary to give a complete logical proof of some theorem.

One teacher, from her experience, has found that the keeping of a note book for the constructions in plane geometry is distinctly valuable. The course of study in plane geometry, as

outlined, requires for the minimum essentials a certain definite number of constructions. As soon as a pupil mastered one of the constructions he added it to his note book which stood as an evidence of the accomplishment of this unit of work.

Two of the teachers in the first semester of plane geometry suggested to the pupils an "exhibit" as one of their projects. The exhibit took the form of posters which were made by the members of the class. As a source of material for the posters, the pupils had access to the advertisements in the current periodicals and magazines. The object of each poster was to display some advertisement in which the pupil had listed a large number of geometrical figures. The posters when completed were hung on the walls of the room. One day, the recitation period was used to judge and score the posters. Each member of the class scored all the posters, save his own. The pupil whose poster received the highest score was declared the winner of the exhibit. Of course, the object of the whole project was to arouse the pupils' interest in geometric form and to make them keenly aware of its use in daily life.

In the writer's classes in plane geometry, two special construction plates are required of each pupil during the second semester. One is an architectural design, usually a cathedral window, and the other, a tiled floor or linoleum design. To prepare for this, reference books on art and architecture are made available to the pupils. They are encouraged to observe the architectural forms in the city. Since the city has no art gallery worthy of the name or no cathedrals, books, pictures and prints have to be relied upon to stimulate the imagination. This year the contributions were lovely. The pupils took delight in arranging their designs and plates on the walls of the room and in inviting others to see their "art gallery." Some of the rose windows, exquisite in execution and color, almost rivaled the one of Westminster Abbey.

One of the teachers in junior algebra incorporated into the recitation period a constructive "check-up" on the daily assignment. The second year algebra is an elective course, and, as a rule, only those who are interested in mathematics or have some mathematical ability enroll in the course. The pupils usually pursued the same unit of work. The individual differences were taken care of by maximum and minimum assignments.

At the beginning of the recitation period each pupil was required to solve the "class-problem" on the day's assignment. The papers were graded and a record of the grades kept. In this way there was a definite daily "check-up."

It has been found necessary to begin second-year algebra with an all-inclusive review of the fundamental processes of elementary algebra, as a year of plane geometry has intervened since the pupils studied elementary algebra. Two of us found that the timed tests by the authors of a certain well-known text book constituted a most satisfactory review. In one class the tests were given during the first eight minutes of the recitation period; in the other class, sometimes at the beginning of the recitation and sometimes before or after school. The pupils manifested much interest in checking their scores and in finding out how much or how little elementary algebra they knew.

Perhaps the most interesting part of the whole course in mathematics is the eighth semester. The main unit is solid geometry. Since the satisfactory completion of the work in solid geometry does not require a full semester, the extra time is used in ways which are deemed most profitable to the students of the class.

Before discussing the use made of the extra time, it might be of interest to comment upon the models made in solid geometry. Several members of the class constructed regular icosahedrons by soldering wires together. Not satisfied with this, they used finer wire for the altitudes of all the triangular faces, and with various colored strings inscribed the other four regular polyhedrons.

It was remarkable what clever models the pupils made with chalk boxes, thumb tacks, wires and strings. Those who were mechanically inclined made rather pretentious models out of more substantial material.

The teacher who usually teaches this course showed herself to be a real discoverer when she hit upon soap as a splendid medium for making prisms, pyramids, frustums and truncated solids. The evolution of an oblique parallelepiped into a right rectangular parallelepiped was one of the unique results.

During the first semester of the past year, after the class had finished the required unit in solid geometry, time was given to the discussion of mathematical topics and to the study of the slide rule. The members of the class asked for the slide rule,

since nearly all of them were intending to enter scientific courses in college.

In the second semester, a part of the extra time was used for review and tying up some of the loose ends, for instance, there was a review unit on the theory of equations, including graphical solutions, another unit on the review of the fundamental formulas in trigonometry, and still others on topics which the student called for.

The remainder of the extra time was used in the discussion of topics of mathematical and scientific nature. Each member of the class was held responsible for the discussion of some subject. Among the topics discussed were the following: "The Evolution of the Calendar," "The Development of π," "The Number System," "Magic Squares," "Relativity," "Television," "Snow Flakes," "Bureau of Standards," "Trade and Barter," "Puzzles and Paradoxes," the lives of a number of mathematicians and many historical facts as revealed in the pages of Smith's "History of Mathematics." The last day closed with an enthusiastic debate on "Square versus Circle."

These talks and discussions, as nothing else, aroused the general interest in mathematics. A few of the talks which were especially illuminating were repeated to other classes. An indirect aim was to make the seniors missionaries, as it were, to advertise the "human appeal" of mathematics. One of the talks which had the strongest emotional appeal was given by a member of the class to a group of sophomores on "What My Course in Mathematics Has Meant to Me." He is a boy whose life's interest seems to be toward law or commerce, yet he chose to study mathematics for four years. Without any suggestions from the teacher he prepared his talk, and in a clear, forceful way enumerated the various helps and benefits he had received from his course in mathematics. It is likely that he had not heard of the recent blow that psychologists have given to the doctrine of transfer of training, but in his young enthusiasm he attributed much to his four years of mathematical study. Let us hope that he had received a little more from his course in mathematics than the knowledge of a few facts and the acquiring of a few skills. Who knows, but that he had a glimpse into that field of human knowledge, in which so much of the thinking of the world has been done.

TWO VARIABLES[1]

By ALICE W. FARRAR

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The two variables about which I wish to speak are skill and understanding. It is very difficult to separate them sufficiently to solve for either in terms of the other, but I feel quite sure that the problem of our students' success in algebra can be satisfactorily solved only in terms of both.

In his chapter on habit, William James says, "The great thing in all our education is to make our nervous system our ally instead of our enemy. It is to fund and capitalize our acquisitions, and live at ease upon the interest of the fund. For this we must make automatic and habitual as early as possible as many useful actions as we can, and as carefully guard against the growing into ways that are likely to be disadvantageous. The more of the details of our daily life we can hand over to the effortless custody of automatism, the more our higher powers of mind will be set free for their own proper work." Let us say, then, that skill in the handling of any process has been attained when we can do the work so quickly and easily that our higher powers of mind are not brought into it at all. Without such skill much of the work which must be done by those who are going into occupations where they will make constant use of mathematics will be unendurably laborious; while those who will not give much thought to mathematics beyond their school days will miss all the pleasure of the study. I trust I am not heretical in this utilitarian age in implying that the pleasure to be derived from studying mathematics is one good reason for including it in the curriculum.

I doubt, however, if any skill not founded on understanding will long remain at the level of a skill, or at least of a very effective one. Recently one of our boys back from college complained that while he got high marks in calculus, the whole thing was not much more to him than a series of rather meaningless "stunts." When he spoke of this difficulty to his in-

[1] Read before The Association of Teachers of Mathematics in New England at their Spring Meeting at Boston University, April 30, 1927.

structor, he was told not to worry, for the meaning of it would probably "dawn on him" before the end of the year. Now this particular boy had attained considerable skill without much understanding. Since he realized the lack himself and was constantly on the lookout for the "dawn" of the main idea, he was in no great danger of remaining forever in darkness. Ordinarily, however, without this fundamental understanding of what it is all about, even a high degree of skill may easily vanish in the short space of a summer vacation. I sometimes wonder if it may not be because our skills have not been firmly rooted in understanding that we often find that our pupils in the eleventh year have lost so much of the ground which they gained in algebra in the ninth grade.

On the other hand, there is the chance that one may understand a process, perhaps even some process which one may need frequently to perform, without being able to do it automatically. Because I had observed that $9 + 7$ was a combination over which my pupils often hesitated, I thought it would be interesting to see if the same difficulty existed among young children. When I asked a nine-year-old girl to add 9 and 7, it was very apparent that she had to stop and think. After she had given the correct sum, I inquired how she had thought it out. "Well," she said, "if you take 1 from 9 and put it with 7 you will have two eights, and everyone knows that two eights make sixteen." Later I put the question to a boy of the same age. After much expression of scorn for the ignorant person who could not do so simple an example, all of which gave him time to think it out, he too produced the right result, and when pressed for his method was ready with an even better one: "If you take 1 from the 7 and put it with the 9, you will have 6 and 10, and I should think you could see that 10 and 6 make 16." Frankly I was surprised at this reasoning, for I had expected that they would know the answer mechanically or else simply count up on their fingers. But of course it was not enough if instruction stopped there. These children should be taught to fund and capitalize this acquisition in a way which would make it possible for them to live at greater ease upon the interest.

I think such is the case with all the fundamental operations of arithmetic. It is difficult for a child really to understand algebra, let alone enjoy it and make satisfactory progress, if in

addition to all the new points to which he should be able to give his undivided attention his mind must constantly shift from the main issue of the problem to the laborious performance of some "task which ought hardly to exist for his consciousness at all."

As we teachers in secondary schools feel that we have a right to expect our pupils to come to us more or less skilled in the handling of numbers, I think we should hold ourselves responsible for developing a like skill in certain frequently recurring processes of algebra. Take for one example the subject of directed numbers. There is considerable question as to the point in the first-year work when this topic should be introduced. There are some writers, notably the Englishman, T. P. Nunn, who believe that the pupil should not have put upon him the extra burden of work with positive and negative numbers until he has had long experience in algebra independent of them. The reason seems to be that the notion is so new to the beginner that to expect him to understand it and apply it in new situations is inviting trouble at the start.

Of course we all recognize that errors in sign are among the commonest that occur. But is it because pupils do not understand what the correct sign is? I think not. It does not seem to me especially difficult to make the laws of signs appear reasonable even to rather ordinary pupils. When required to justify their choice of sign they can usually do so. Rather the difficulty is apparently that in processes where for seven or eight years they have been accustomed to handle numbers without reference to sign at all, they will not now hold themselves in check long enough to give sufficient consideration to the matter. I try to encourage pupils to form the habit of thinking sign first, then number. It is the natural order in writing; it is the most important thing, since it places the number in its class; and being the latest feature learned about a number, it is most likely to be the point of difficulty and therefore should be attacked first. It is the pupils who are still probably painfully conscious of effort in making number combinations in arithmetic who are more likely to make errors in sign. Since I feel that the trouble with signed numbers is due not so much to lack of understanding as to carelessness, I believe that the topic should not be postponed until so late in the year as Nunn and some of his followers suggested, for thereby the student will miss too many opportunities for over-

coming his tendency to be careless as well as much practice leading to the acquisition of an essential skill. In fact it might happen that in many cases the skill would not be thoroughly established before the student would drop all thought of algebra for at least the summer vacation and perhaps for an entire year.

Not only should a student of algebra feel himself on firm ground in handling the fundamental processes with integers but with fractions as well. The weakness of most pupils in all work with fractions is due, I believe, to lack of understanding. When first taught, perhaps the various operations are thoroughly rationalized for all but the poorest in the class. Then comes the process of bringing the whole matter up to the level of skill. Later as concentration of attention is shifted to other topics, there is a loss of ability to deal with fractions, and when it is revived it is revived merely as a skill without first going back to the underlying principles.

It might be somewhat difficult to justify all the time we spend upon algebraic fractions were it not for the fact that we must go back and attempt once more to make clear the very simple principles which are at the foundation of all the work. We might expect to use the pupils' knowledge of fractions in arithmetic to illuminate the work in algebra. Of course, there are many instances where this can be done, but not infrequently it is the work in algebra which throws much needed light on the corresponding operations in arithmetic. Merely because the manipulation of algebraic fractions is more complicated than work with the simple ones which now occur in most arithmetics, the process itself with its meaning is much more apparent.

In either arithmetic or algebra, I believe that any method which constantly emphasizes the reasons involved is to be preferred to a special device, short cut, or form of arranging the work which, though it may make for speed and even accuracy at first, yet casts a mist over the fundamental principle used. It is in this connection that one finds the advantage, to cite just one example, of teaching square root by the estimate-and-try method. Suppose we admit that the other way is shorter and that it is not impossible to explain to good students by reference either to algebra or geometry the reasons for the steps taken. Still it is hardly worth devoting the necessary time to develop skill in handling this process to the point where it can long be re-

tained, for those who need frequently to find square root will use tables or a slide rule. With ordinary teaching the others are almost sure to forget the process in a year or two. The guess-and-try method, just because it depends so obviously on the very meaning of square root, will not easily be forgotten, and the fact that it may take a little longer, even if contracted methods of dividing are used, is not an important consideration in a process which one seldom uses.

If, on the other hand, there are two methods equally reasonable, it would seem wise to choose only the one which gives the fewest opportunities for making errors and which will be most concise and rapid where in future the student of mathematics will need to make use of it. As an illustration of what I mean, consider the two methods of simplifying a complex fraction which are given in most textbooks. The one usually mentioned first looks upon a fraction as indicated division and suggests that it be treated as such. The second method depends upon the principle that multiplying both numerator and denominator by the same quantity does not change the value of a fraction. There is certainly no need of studying both methods. In either case, there is no great difference in the importance of the principle involved. If required to think out a method with no assistance, perhaps there is a slight chance in favor of a pupil's seeing for himself the first method, though this would probably not be the case if he had never done so in arithmetic. As for ease in all cases where he is likely to make use of complex fractions in future, there is everything to be said in favor of the second method. What are those cases? For the first year the only situations where the student will meet complex fractions are in checking fractional equations the roots of which are not integral and in substituting in a few formulas. In more advanced work fractions in which terms have negative exponents can be reduced more readily by multiplying both numerator and denominator by the proper quantity. Later still, in proving trigonometric identities this second method shows its superiority as far as ease and compactness are concerned.

There has been so much said at the meetings of this association in favor of handling all equations with direct reference to the axioms that the point hardly seems worth mentioning today. It is one of those instances where in the beginning speed and pos-

sibly even accuracy for the poorer pupils must be sacrificed in favor of a rational notion of the whole subject. But I believe that this postponement in the acquisition of skill is not worth considering in face of the gain in power and understanding that can function in later work. For after all, the greatest stumbling block in the way of the child's progress in all mathematics is the feeling on his part that he can get along if he can only learn to perform certain tricks. This is an idea firmly rooted in the minds of many of our young students for whom mathematics is naturally difficult, and an idea that is constantly strengthened by their observation of the work of their quicker classmates. They simply cannot believe that the examples can be done so easily without the aid of some more or less mechanical device or trick. How significant is the remark often made by such pupils, "I didn't have any luck with today's work." But we cannot make much progress until we can succeed in exploding completely this fixed notion that luck enters into it at all; for only then will they cease expending the major part of their energy in a frantic search for a scheme that will enable them to put down on paper statements which for some reason will please their teacher and so earn for them the desired mark.

The long time early in the year which we used to devote to teaching factoring was calculated to impress all but the good pupils with the importance of learning certain tricks of the trade. I believe that we can still further reduce this evil in the case of both special products and factoring if, instead of beginning with the square of a binomial, we consider the general type first, as is done in a few recent texts. The pupil may ask, "Why take it at all?" He already knows how to get the result in one way. Why isn't that enough? The teacher might answer for the same reason that after learning to add by putting similar terms in the same column, we spent most of our practice on examples in which all the terms were written on a line. That is, simply because ninety-nine times out of a hundred the work will appear in that form, and we might as well learn to save ourselves the trouble of rearrangement. Just so with multiplication of this sort which we seldom need to do except as a part of some longer problem. If we can see how to write the result at once without doing it out as a separate example on another part of the paper, it will save time.

Included in the practice exercise which would naturally follow this discussion, there should be many examples of the special types, the square of a binomial and the product of the sum and difference of two numbers. The pupils themselves can be counted upon to comment on the fact that $(a + b)(a - b)$ represents a special case in which the work is very much reduced if they but learn to recognize it. $(a + b)^2$ will perhaps seem especially difficult because of the apparent necessity of rewriting in the form $(a + b)(a + b)$. Then they may readily be led to observe that by learning the special facts that apply here, this too will be a type simpler to do than is the general case. The quality of the class will then determine how much time must be spent in developing skill in handling these special types of the general form, but I have found that the whole ground can be covered in less time if the work is taken in this way.

All factoring which we need to teach in the first year can then come under two cases: (1) an expression whose terms have a common monomial factor and (2) an expression which is the product of two similar binomials, that is, the reverse of the special product taught. Of course students should learn the great advantage of recognizing a perfect trinomial square and the difference between two squares.

A few weeks ago a boy who had been in my class last year came back from college and, full of enthusiasm, said, "Really, calculus is the important part of mathematics. Geometry was valuable because of the way we learned to think, but while I liked to do algebra, I didn't believe it was really worth much and I still do not think it would be except that you need to know most of it for calculus. I remember last year in geometry you spoke of the importance of calculus, but you didn't make it very clear, so I paid no attention. If from the start pupils only knew how important and interesting it is, the rest of the work would have more meaning so that they would do better. Maybe it might be hard to make them really understand," he added. About as difficult, I thought, as trying to tell a person who had never heard any music what he would find enjoyable in a symphony concert. Yet the boy was perfectly right in the underlying idea; for it is only as we have some understanding of purpose in the whole that we can become truly interested in the parts. To be sure, there are those who will like work in

mathematics simply because they enjoy mental activity, but mere going for the sake of being on one's way, though one knows not whither, is not enough for most of us.

If our pupils could live at ease upon the interest of their teachers' investments, it might be sufficient for us merely to tell them that their present work is worth doing because of the advantage to be derived from more advanced study in the future. But in the meantime they want to be aware of some present worth. Can they not be taught to appreciate even in the earliest stages that algebra is a powerful instrument or tool for solving problems that have to do with quantity? All problems, whether mathematical or not, are questions of relationship, and their solution consists of making exact statements of those relationships. One of the chief impressions which the student should gain from a course in algebra is a vivid sense of the power of algebra in solving problems—special problems reduced to solvable equations and generalized problems expressed in formulas.

This may be easier said than done, as my experience only a week or two ago will illustrate. I thought that everyone in my class understood this point. We had been doing problems based on the law of the lever and had spoken of the lever as an instrument or tool which enables man to accomplish things which he would not otherwise have sufficient power to do. As we were then about to take up a new type of equation, I thought I should be making a fine transition if I asked, "What powerful tool do we have in algebra?" The immediate response was "The equation," just as I expected. So I continued, "The equation is a tool which we use for what purpose?" How pleased I was when I saw that a boy with an I.Q. of 83 appeared to know and was eager to tell us. I gave him a chance and he said, "The equation is a tool to stir up your brains and make them work." At least I had the satisfaction of seeing that the class as a whole was greatly amused at this reply, and the boy beside him looked so especially scornful that I called on him, only to learn that it is the brain that is a tool for solving equations. However, the proverbial third time did not fail and the next boy informed us that the equation is a tool which helps our brains solve problems which would be too difficult without it.

In all discussion about problems in algebra the question is raised regarding their reality to the pupil. Now it seems to me that this troubles most of us more than our pupils, and that

we can settle the question of *real* algebra problems as our two speakers at the last dinner agreed to settle the matter of *practical* originals in geometry. They must appear real enough to the pupil to interest him, but the appearance of reality does not necessarily depend upon the frequency with which such problems might actually occur in life situations. A problem which is found in Sam Lloyd's *Brain Food* or a scout magazine is of importance in the eyes of a boy no matter how absurd it may be in itself.

Interest depends very largely also upon a proper degree of difficulty. A problem just difficult enough to require effort but not so difficult as to be impossible of independent solution will usually interest, and so seem worth doing. Most writers of textbooks do not give this point sufficient attention. It is only because we all enjoy success and dislike failure that the poor student usually prefers work in more or less mechanical manipulations to verbal problems.

I believe that many pupils of this sort can do simple problems more readily if they are allowed to use two unknowns. For this reason I advocate teaching some sort of solution of simultaneous equations very early in the year, certainly before factoring. Usually the occasion arises naturally when in considering a very simple problem such as The sum of two numbers is 18 and their difference is 4, a really dull pupil suggests letting x represent one number and y the other. He will then perhaps write the equation $x + y = 18$ and be unable to solve it. If he has already been introduced to the graph, he can plot the line represented by this equation and see that it alone is not sufficient to solve his problem. Then it is possible to get the other equation and, by plotting the two, find the one solution at their point of intersection. It may happen that some bright pupil in the class will see how by using the method of substitution we can get from these two equations the same one that he wrote using only one variable. I never knew anyone to work out unassisted the solution of such equations eliminating by addition or subtraction, although this is the method which most pupils prefer and use more efficiently when it is called to their attention. Teaching simultaneous equations thus early not only helps the dull students in solving problems but gives opportunity for additional drill in the use of directed numbers and adds point to the study of the graph.

While problems are preeminently a matter of clear understanding, it does seem as if we might raise almost to the level of a skill the tendency to immediate action in listing and expressing the various elements involved. In the same paragraph from which I quoted in beginning, James says: "There is no more miserable human being than one in whom nothing is habitual but indecision and for whom the beginning of every bit of work is a subject of express volitional deliberation." That exactly describes the apparent state of mind of many pupils when faced with a verbal problem.

A few days ago our school carpenter brought me a problem which his father had been working on in his spare time for two weeks. It could be done in a few minutes by one who knew a little algebra and how to find the area of a trapezoid. He wanted to know how I did it, so that he could tell his father. I suspected that the father had used a guess-try-fail-and-try-again method, which proved to be the case. The man knew how to find the area of a trapezoid but no algebra, so the best I could do was to explain that instead of guessing a number, I used a letter to stand for it and then worked with that letter just as his father had with his estimated number. In that way, because I knew certain rules for handling the result, I could get from my statement the correct answer. As he saw me write down the equation he exclaimed, "Well, if that isn't the neatest idea!" May not one reason that so many find difficulty with verbal problems be that they have never had this profound personal conviction of the tremendous neatness of the method?

Two important aims of a beginners' course in mathematics are the development of a problem-solving attitude of mind and the feeling that the whole system is thoroughly understandable. Time spent in building up mechanical skill before the principle is well understood is usually worse than wasted, but progress toward the goal will be slow and the work irksome unless we do succeed in developing upon a foundation of understanding a degree of skill in the performance of the most frequently recurring processes which will make it possible to hand over this work "to the effortless custody of automatism." This is, I presume, what we all try to do, but I must confess that like Portia, "I can easier tell twenty what is good to be done, than be one of twenty to follow mine own teaching."

HOW ALICE MADE PI MU EPSILON

By PROFESSOR JAMES B. SHAW

University of Illinois

Alice had learned in her adventures in Wonderland that almost everything was just a game, and could have been played some other way. In her trip through the Looking-glass she had found that things in the world went along about as well backwards as they did forwards. She had developed a rather cynical attitude as to whether anything was certain, and could be relied upon, until she went to High School and studied Algebra and Geometry. In the High School and the Junior College she had come to have a very high regard for Mathematics, the only subject in which one knows what he is talking about and whether what he says is true. She made only A grades, and was not much surprised one day to receive an invitation to a party, which could mean only that she was being rushed for Pi Mu Epsilon. She met a number of her friends there, and did a fair share of the puzzles she was to solve. Evidently she made a good impression, as later she received an invitation to join this honorable society. As part of the initiation, after she had secured the signatures of most of the department, she was to go to the Brownfield woods, and meet the White Rabbit who would conduct her examination. The dormice and the caterpillar were to serve as committee.

At the appointed time Alice was punctual, but as usual the committee were fully half an hour late. The White Rabbit finally sauntered in, and glancing ironically at her said:

"It's you, is it?"

"Yes," said Alice, a little put out at not being congratulated on her presence. The White Rabbit sat down, and after a while said:

"Do you know Arithmetic?"

"Of course I do," said Alice, a little indignant at the absurdity of the question.

"Well then," said the Rabbit, "perhaps you can answer this question? If a man sells a horse for fifty dollars, then buys it

back the next day for forty-five, and sells on the following day for sixty, how much does he make?"

"Twenty dollars," said Alice.

"Wrong," said the Rabbit, "he lost twenty dollars, for the horse had cost him eighty-five."

"But you didn't say that at first," said Alice.

"That doesn't matter. You must always guess what the examiner is thinking of," said the White Rabbit.

"Perhaps you can answer this one better," said the Rabbit. "If you have a glass half full of water, and another half full of wine, and transfer a teaspoonful of the wine to the glass with water, then, after stirring, transfer a teaspoonful back to the first glass, is there more water in the first glass than wine in the second?"

"No," said Alice, "there is less, for you take out a teaspoonful of wine but you put back a teaspoonful that isn't quite all water."

"Wrong again," said the Rabbit. "The amounts are just equal.

"Do you understand the Rule of Three?"

"Yes," said Alice, "I used to like that in school."

"What is the principle of the rule?"

"The greater is to the less as the greater is to the less."

"Indeed!" said the Rabbit, looking at her severely over his spectacles.

"I suppose you will admit that 8 is greater than -4, and 3 is greater than -6."

"Yes," said Alice, beginning to feel queer.

"Well," said the Rabbit,

$$"8:-4=-6:3$$

and the greater is to the less as the less is to the greater. You are wrong again."

By this time Alice was almost in despair, for the questions took such unexpected turns they quite upset her. She began to think she was not likely to answer any of his questions.

"What is a number?" said the Rabbit.

Alice thought for some time over this question, and finally concluded she had better show off the knowledge she had acquired in one of her courses, so she said, "If you can match two aggregates, member to member, they are said to have the same number."

"Very well," said the Rabbit, "there came to a hotel one night seven men, each of whom insisted upon having a room to himself. The inn-keeper had only six rooms, but he did not believe much in Arithmetic, so he put the first man in a room, and then after thinking a while, asked another one to wait there for him. He then put the third in the second room, the fourth in the third room, the fifth in the fourth room, the sixth in the fifth room, went back and got the seventh whom he put in the sixth room, and everybody was satisfied." Alice was a little dazed after this but she thought the best thing was to say nothing.

"Do you know Algebra?" said the Rabbit.

"Yes, I liked it in the High School," said Alice.

"What is one of the axioms of Algebra?"

"Equal multiples of equals are equal," said Alice, selecting the first she remembered.

"Indeed!" said the Rabbit. "You will admit that a bottle half-full is a bottle half-empty."

"Yes," said Alice, wondering what was coming now.

"Well, multiply both sides by two, and a bottle full is a bottle empty," said the Rabbit.

"Well, it is that way in the United States anyhow," said Alice, a little impudently.

"How much is $(-1)^2$?" said the Rabbit.

"It equals $+1$."

"And $(+1)^2$?"

"That is $+1$ also," said Alice, sure of herself this time.

"Then $(-1)^2 = (+1)^2$?"

"Yes," said Alice.

"Therefore $\log (-1)^2 = \log (+1)^2$?"

"Yes," said Alice.

"Well then, $2 \log (-1) = 2 \log (+1) = 0$?"

"Yes," said Alice doubtfully.

"Therefore $\log (-1) = 0$ and $-1 = +1$?" said the Rabbit.

"Oh! Dear!" said Alice, "I have got it all wrong again."

"Do you know Geometry?" said the Rabbit.

"I used to," said Alice weakly, for she was getting cautious and discouraged.

"What is the shortest line between two points?" said the Rabbit.

"A straight line," said Alice.

"Wrong again," said the Rabbit. "A telegraph line is the shortest between two points."

"I didn't know you meant that," said Alice indignantly.

"In Mathematics," said the Rabbit, "we are likely to mean most anything. I will ask you another question. You ought to know this one at least. How do you find the hypotenuse when you know the sides of a right-angled triangle?"

"You square the sides, add the results, and take the square root," said Alice rather confidently.

"Is the result more than the sum of the sides?"

"No," said Alice, "it is less. It must be that way in any triangle—"

"Wrong again," said the Rabbit, "for if you proceed along one of the isotropic lines that go to the circular points, the distance from any point to any other point is zero, and the line is perpendicular to itself, so you can have a right angle anywhere, and all the lines are zero."

"Who ever heard of a line perpendicular to itself?" said Alice.

"You did, just now," said the Rabbit severely. And Alice was silent.

"How long is a semicircle?"

"It is 1.5708 times the diameter," said Alice.

"If you draw a semicircle and its diameter, and then on each half of the diameter construct a semicircle, one above, and one below, how long is this curve which is shaped like a letter S?"

"As long as the semicircle," said Alice, after thinking a little.

"Well," said the Rabbit, "if now you bisect each of these diameters and repeat the construction of semicircles, and do this indefinitely, the wave line you get is always just as long as the semicircle, but it approaches as close as you please to the diameter, so the length of the semicircle has as limit the length of the diameter."

Alice was now almost in tears, for all she had thought she knew was crumbling before her very eyes.

"Do you know anything about Calculus?" said the Rabbit.

"Yes," said Alice doubtfully.

"What is a second derivative?"

"The derivative of the first derivative."

"Can there be a second derivative without a first?"

"Certainly not."

"Wrong," said the Rabbit. "There are cases of second derivative but no first."

"How do you find a maximum or minimum?"

"You differentiate and set the result equal to zero," said Alice.

"Very well," said the Rabbit. "Find the maximum radius of an ellipse. The formula is $r = a + ex$."

Alice differentiated and got the derivative equal to e, which she knew could not be zero, so she replied rather hesitatingly: "It doesn't seem to have any."

"Then they are all alike," said the Rabbit, "and an ellipse is a circle. You are wrong again. Can you integrate?"

"A little," said Alice.

"What is the integral of $1/x$?"

"$\log x$," said Alice.

Integrate $1/x$ by parts," said the Rabbit. Alice did so and got $1 + \log x$.

"That proves $1 + \log x = \log x$," said the Rabbit, and therefore $1 = 0$.

Alice merely sighed.

"What is the linear continuum?" said the Rabbit.

"It is the set of all real numbers," said Alice, reciting her definition from Real Variables.

"How far apart are the numbers?" said the Rabbit.

"I don't know what you mean," said Alice.

"If you put them as points along a line, how big will the gaps be?"

"There won't be any gaps," said Alice.

"Wrong again," said the Rabbit. "They may have gaps you could drive a bus through."

"You have flunked every question," said the Rabbit impatiently. "If you are going to be initiated at all, you will have to answer the last one correctly. What is Mathematics?"

Alice puzzled over this some time, and at last in sheer weariness could think of nothing but a quotation, so she said: "Mathematics is a sublimated form of play, the austere and lofty analogue of the kitten with the tangled skein, or the young eaglet sporting with the mountain wind." So she was initiated, and began to play.

NEWS NOTES

THROUGH the leadership of Professor S. T. Sanders of the University of Louisiana, a section of the National Council of Teachers of Mathematics has been organized to assist in the improvement of mathematics in Louisiana and Mississippi.

THE spring meeting of the Philadelphia Section of the Association of Teachers of Mathematics in the Middle States and Maryland was held in Overbrook High School, Saturday, March 19, 1927, at 9:30 A.M.

The following officers were elected for the ensuing year:

President, Muriel Smith, Overbrook High School.
Vice-President, Clarence A. Garbrick, Central High School.
Secretary-Treasurer, Julia M. Bligh, South Philadelphia High School for Girls.
Member of Executive Committee, Frederick H. Safford, University of Pennsylvania.

Following the business meeting Miss Marie Gugle, Assistant Superintendent of Schools, Columbus, Ohio, and President of the National Council of Mathematics Teachers, gave her illustrated lecture on "Dynamic Symmetry."

WALTER C. EELLS, member of the Advisory Board of the MATHEMATICS TEACHER, who for the past eleven years has been professor of applied mathematics at Whitman College, has resigned to become associate professor of education at Stanford University.

THE LOUISIANA-MISSISSIPPI SECTION OF THE MATHEMATICAL ASSOCIATION OF AMERICA

ONE hundred mathematics teachers and mathematics supporters sitting down to a dinner in Hotel Youree, Shreveport, Friday evening, March 4, marked the highest level of mathematical interest yet attained in the history of Mississippi and Louisiana. Centenary College and Shreveport Chamber of Commerce did the part of joint hosts to the mathematical visitors superbly. The entire mathematics constituency of the La.-Miss. Section is indebted to President Sexton, Dean Hardin, Professors Maizlish and Reynolds, to Mayor Thomas, Mr. Yandel Boatner and those assisting them for the effective way in which they contributed to the success of the third annual meeting of the Louisiana-Mississippi Section of the M. A. of A.

The speech of President Boatner showed, as was to be expected, keen appreciation of the part mathematics plays in the development of industrial and scientific projects.

As Planned

Things moved as by previous arrangement, with almost clock-like precision. Yet, flexibility of program, social contacts and sight-seeing were

elements not wanting to the occasion. The round-table discussion of Friday evening was fruitful of much that is significant. With the assistance of Miss Marie Gugle, president of the National Council of Teachers of Mathematics, who was present, a Louisiana-Mississippi branch of the Council was organized and officers were duly elected as follows: W. C. Roaten, DeRidder, La., Chairman; Miss Ida K. Smith, Meridian, Miss., Vice-Chairman; Miss Norma Touchstone, Secretary-Treasurer.

Acting jointly, the Section and the Council voted to hold their annual meetings at the same time and in the same place, the annual programs to be made out by the officials of the two organizations.

In this manner has been brought about a close official articulation of the high school and the college mathematics teachers of the two States. Properly used and steadily developed, such organic association of the two groups of mathematicians is bound to have a transforming influence on the mathematics teaching of this territory and, ultimately, to be a powerful factor in deepening the public interest in mathematics, and in bringing about a more adequate rating of the value of mathematics in every scheme of education and training for citizenship. *For mathematics* (a) *is the fundamental underpinning of every science,* (b) *is being increasingly appropriated as a necessary instrument in the so-called inexact sciences,* (c) *is an unequalled means of habituating a mind to well-ordered logical thinking, and to precise analysis.*

But this Articulation of the Two Mathematical Groups Can be Maintained Only at the Cost of Effort!

The benefits are not to be realized unless certain conditions are met. Effort and struggle must be steadily put forth by an increasing number, if they are to be attained. Estimating that approximately two thousand people are teaching mathematics in the two States, it is a safe assertion that sufficiently rapid growth of the influence of the Mathematical Association of America and of the Council is not being maintained in this territory unless something like the following results are reached:

(*a*) Each week should add several new mathematics teachers to the roll of membership of Louisiana-Mississippi chapter of the Council.

(*b*) At least one new member should be added to the M. A. of A. from La.-Miss. territory every week of the year on an average.

(*c*) In each issue of the bi-monthly news letter there should be several communications from mathematics teachers, college or high school communications aiming to stimulate mathematical interest anew, to impart valuable information of a mathematical sort, or to propose some question of mathematical interest.

(*d*) Steadily, throughout the year, there should be sent into our offices, to be added to our mailing lists, the names of mathematics teachers or mathematically interested persons.

(*e*) Money—absolutely essential to carry out every part of the Section plans—should be made available to the officials without the necessity of undue expenditure of effort. Schools, colleges, universities, mathematicians are urged to make generous contributions to the maintenance of this pro-

gram of mathematical development in our two States, *for there is no form of university or high school or college extension work more worthy of support than the cause of mathematics and mathematics teaching!*

S. T. SAUNDERS, *Chairman*,
Baton Rouge, La.

THE Association of Teachers of Mathematics in New England held its spring meeting at Boston University, Saturday, April 30, 1927. The program included: (1) Two Independent Variables, by Miss Alice W. Farrar, Brockton High School; (2) Present Day Teaching of Mathematics, by F. Eugene Seymour, Superintendent of Mathematics for the State of New York; (3) Certain Topics in Algebra, by Heinrich Wilhelm Brinkmann, Harvard University; and (4) New Type Test in Mathematics, by Professor William D. Reeve, Teachers College, Columbia University.

The officers are: William S. Vosburgh, Teachers College, Boston, President; Professor Raymond K. Morley, Worcester Polytechnic Institute, Vice-President; Harry D. Gaylord, Browne & Nichols School, Cambridge, 20 Charlesgate West, Boston, Secretary; and Harold B. Garland, High School of Commerce, Boston, Treasurer.

Psychological Analysis of the Fundamentals of Arithmetic. By CHARLES HUBBARD JUDD. The University of Chicago, Chicago, Illinois. Pp. x + 122. $1.00 postpaid.

The investigations reported in this monograph deal with the mental processes of children and adults when they are counting or making the simpler number combinations. The mental processes involved in elementary number exercises are detailed with a precision achieved by no other investigator.

Counting, as well as addition, subtraction, and other forms of arithmetical thinking, is found to be a complex organized pattern of behavior. Instruction in arithmetic cannot properly be organized without full regard for the complexity of these processes. Appreciation of quantitative thinking comes only through the extended use of numbers. One remedy for ineffective teaching of arithmetic is to be sought in new and more effective methods of introducing pupils to our number system.

Authors of textbooks in arithmetic and professional books on the teaching of arithmetic should consider most carefully the findings and recommendations appearing in this monograph.

ANNOUNCEMENT

Beginning with this issue THE MATHEMATICS TEACHER will be published under the combined editorship and business management of Dr. John R. Clark of the School of Education at New York University and Dr. W. D. Reeve of Teachers College, Columbia University. All communications relating to THE MATHEMATICS TEACHER should be addressed to THE MATHEMATICS TEACHER, 525 West 120th Street, New York City.

The editors are desirous of having all members of the National Council participate as fully as possible in realizing the following aims:

(1) To create and maintain interest in the teaching of mathematics.
(2) To keep the values of mathematics before the educational world.
(3) To help the young and inexperienced teacher to become a good teacher.
(4) To improve teachers in service by making them better teachers.
(5) To raise the general level of instruction in mathematics.

In order to realize these aims we offer the following suggestions:

1. That teachers of mathematics everywhere form themselves into local groups—city, sectional, or state—these groups to be closely connected with the national organization for a cooperative and thorough study of the problems in the teaching of mathematics.

2. That each individual teacher become a member of the National Council. This may be done by sending two dollars to THE MATHEMATICS TEACHER at the address given above. This fee entitles each member to receive the magazine for one year without extra charge.

3. That members of the Council assist the editors in obtaining more subscribers by carrying on campaigns for membership in various local associations all over the country. Pamphlets describing the Council and containing subscription blanks, to be

distributed at such meetings will be mailed to those who are willing to cooperate in this work. A postal card indicating such willingness will be sufficient.

4. That members be alert to obtain and sent to the editors good papers for publication in THE MATHEMATICS TEACHER. In lieu of obtaining the papers notify the editors of any good prospects. Members may assist greatly by notifying them of prospective papers.

5. That teachers of mathematics, especially the secretaries of local groups, send THE MATHEMATICS TEACHER notices of meetings that are to be held or which may have been held, including programs of such meetings. Brief comments on such programs will be appreciated.

6. That individuals cooperate by sending THE MATHEMATICS TEACHER news items concerning promotions or changes in position of teachers, new books in preparation, curriculum revisions and the like.

In any or in all of the above ways the members of the Council can help the editors greatly by their interest and cooperation. There is no reason why such a pooling of efforts should not lead in three or four years to a doubling of our present membership.

The Editors.

Arithmetic for Teacher-Training Classes

By E. H. Taylor, Ph.D.

Prepared especially for prospective arithmetic teachers, to give them a new view, the teacher's view, of the subject as a whole.

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